THE WALLACE PROBLEM AND COUNTABLY COMPACT TORSION-FREE ABELIAN GROUPS IN ZFC

JULIANE TRIANON FRAGA AND VINICIUS DE OLIVEIRA RODRIGUES

Abstract. We prove in ZFC that every torsion-free Abelian group of cardinality $\mathfrak{c}$ admits a Hausdorff countably compact group topology without nontrivial convergent sequences. In particular, this applies to the free Abelian group $\mathbb{Z}^{(\mathfrak{c})}$, the Baer–Specker group $\mathbb{Z}^{\omega}$ and $\mathbb{Q}^{(\mathfrak{c})}$. For the topology constructed on $\mathbb{Z}^{(\mathfrak{c})}$, the coordinatewise nonnegative cone is countably compact in the subspace topology. Consequently, there exists in ZFC a commutative Tychonoff countably compact topological semigroup which has two-sided cancellation but is not a group, giving a negative answer to Wallace's question. Combined with earlier results, the main theorem also yields in ZFC a Tychonoff countably compact topological semigroup containing a copy of the bicyclic semigroup and a functionally Hausdorff countably compact paratopological group that is not a topological group.

1. Introduction

A classical theorem of Gelbaum, Kalisch, and Olmsted and of Numakura states that every compact Hausdorff topological semigroup with two-sided cancellation is a topological group [14, 23]. At the 1953 annual meeting of the American Mathematical Society, A. D. Wallace observed that it was not known whether compactness could be replaced by countable compactness. He recorded the question in print in 1955 [35, p. 101]; it was later listed as Question 3L.1 by Comfort in *Open Problems in Topology* [8]:

> Must every Hausdorff countably compact topological semigroup with two-sided cancellation be a topological group?

A negative example is usually called a *Wallace semigroup*. The problem is delicate because several natural strengthenings of countable compactness do force the compact conclusion. For example, the conclusion holds under first countability and under sequential compactness [22, 15]. On the other hand, by assuming additional set-theoretic hypotheses, several consistent constructions of Wallace semigroups have been obtained. Robbie and Svetlichny constructed a Wallace semigroup under the Continuum Hypothesis, and Tomita obtained one under Martin's Axiom for countable partial orders [25, 30]. However,

2020 *Mathematics Subject Classification.* Primary 22A05, 54H11, 54D20, 22A15; Secondary 20K20, 54A20.

Key words and phrases. countably compactness, topological groups, torsion-free Abelian groups,nontrivial convergent sequence, Wallace semigroup.

the question of whether a Wallace semigroup exists in the standard Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) remained open for 73 years.

The route from topological groups to Wallace semigroups passes through a second long-standing problem. A countably compact Boolean group without nontrivial convergent sequences was constructed in ZFC by Hrušák, van Mill, Ramos-García, and Shelah [18]. That construction resolved the general existence problem posed by van Douwen and, together with the reductions of van Douwen and Tomita, the product problem of Comfort [34, 31]. However, it did not yield a Wallace semigroup. Accordingly, Hrušák, van Mill, Ramos-García, and Shelah asked whether there exists in ZFC a non-torsion countably compact topological group without nontrivial convergent sequences [18, Questions 5.5 and 5.6]; see also [17, Questions 19 and 20].

Free Abelian groups form the canonical torsion-free test case. In 1990, Comfort asked whether the free Abelian group on $\mathfrak{c}$ generators admits a countably compact group topology in ZFC, attributing the question to Tkachenko [8, Question 3C.1]. Tkachenko had constructed such a topology under the Continuum Hypothesis [28]. Later constructions used Martin's Axiom or selective ultrafilters [20, 21, 7]. Stronger examples with prescribed compactness of finite powers were also obtained from additional set-theoretic hypotheses [5, 13]. The ZFC question for the free Abelian group, including the stronger requirement of having no nontrivial convergent sequences, remained open [6, Problem 6.2] and [5, Question 6.2].

Our result treats the entire class of torsion-free Abelian groups of cardinality continuum.

Theorem 1.1. *In ZFC every torsion-free Abelian group of cardinality $\mathfrak{c}$ admits a Hausdorff countably compact group topology in which every convergent sequence is eventually constant.*

Theorem 1.1 constructs in ZFC a torsion-free Abelian countably compact group with no non-trivial convergent sequences. Before this result, the known ZFC constructions produced only torsion groups. It was therefore unknown whether there exists in ZFC a non-torsion countably compact group with no non-trivial convergent sequences. This question was posed in [32, Question 6(a)], [17, Question 20], [18, Question 5.6], and [33, Question 6.8]. Theorem 1.1 answers it affirmatively. Moreover, since the groups constructed here are torsion-free, the theorem also proves the existence in ZFC of a torsion-free countably compact group with no non-trivial convergent sequences, as asked in [6, Problem 1.2].

Several classical groups illustrate the gain in generality. Taking $G = \mathbb{Z}^{(\mathfrak{c})}$ resolves Tkachenko's free Abelian problem in ZFC, with the additional absence of nontrivial convergent sequences. Taking $G = \mathbb{Z}^{\omega}$ answers Question 14.7(i) of Dikranjan and Shakhmatov about the Baer–Specker group [9]. We also have the following consequence for the additive group of real numbers.

Proposition 1.2. *In ZFC the additive group $\mathbb{Q}^{(\mathfrak{c})} \approx \mathbb{R}$ admit Hausdorff countably compact group topologies in which every convergent sequence is eventually constant.*

We apply the construction to $\mathbb{Z}^{(\mathfrak{c})}$ and prove that its nonnegative cone is countably compact in the induced topology. Thus the counterexample is commutative and Tychonoff in addition to satisfying the requirements of the problem.

Corollary 1.3. *In ZFC there exists a commutative Tychonoff countably compact topological semigroup with two-sided cancellation which is not a group.*

This gives a negative answer in ZFC to Wallace's question [35, 8]. It also settles its later ZFC formulations in [27, p. 12], [29, p. 524], [24, Problem 523], [7, Question 5.1(a)], [17, Question 19], [18, Question 5.5], [6, Problem 6.4], [33, Question 6.7], and [5, Question 6.3].

The same theorem has further consequences which are not specializations to a single familiar group. Known results reduce problems about the bicyclic semigroup, countably compact paratopological groups, and monothetic monoids to the existence of a torsion-free Abelian countably compact group without nontrivial convergent sequences. Section 10 records these consequences and a consequence for suitable sets. There we also answer a question of Shakhmatov: whether there exists in ZFC a countably compact free Abelian group in which every infinite closed subset has cardinality at least $\mathfrak{c}$ [27, p. 13]. We prove that the topological group constructed on $\mathbb{Z}^{(\mathfrak{c})}$ has this property. We also delimit questions for which Theorem 1.1 gives a partial answer.

2. Preliminaries and the topological reduction

We write $\mathbb{N} = \{0, 1, 2, \dots\}$, regard each $m \in \mathbb{N}$ as the finite ordinal $\{i : i < m\}$, and index every m-tuple by $i < m$. We use additive notation. The circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ carries the quotient metric induced by the Euclidean metric on $\mathbb{R}$, determined by

$$\|x + \mathbb{Z}\| = \min_{n \in \mathbb{Z}} |x - n| \qquad (x \in \mathbb{R}).$$

If $(x_n : n \in \mathbb{N})$ is a sequence in a Hausdorff topological space and p is an ultrafilter on $\mathbb{N}$, then $p\text{-}\lim_n x_n = x$ means that

$$\text{for every neighborhood } U \text{ of } x,\ \{n : x_n \in U\} \in p.$$

Every ultrafilter on a compact Hausdorff space converges to a unique point. An ultrafilter on $\mathbb{N}$ is free when it contains the cofinite filter.

The circle is divisible and therefore injective in the category of Abelian groups. Consequently a homomorphism from a subgroup of an Abelian group to $\mathbb{T}$ extends to the whole group [12, Theorem 21.1]. We write $\mathrm{Hom}(G, \mathbb{T})$ for the set of group homomorphisms from an Abelian group G to $\mathbb{T}$.

As usual, $\mathbb{Q}^{(\mathfrak{c})} = \{z \in \mathbb{Q}^{\mathfrak{c}} : |\operatorname{supp}(z)| < \infty\}$ denotes the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$, where $\operatorname{supp}(z) = \{\xi < \mathfrak{c} : z(\xi) \neq 0\}$ is the support of z.

We adopt the convention that, if $D \subseteq \mathfrak{c}$, then $\mathbb{Q}^{(D)}$ denotes the subgroup of $\mathbb{Q}^{(\mathfrak{c})}$ consisting of all vectors supported in D. Thus, each element of $\mathbb{Q}^{(D)}$ is a function from $\mathfrak{c}$ into $\mathbb{Q}$ that vanishes outside D.

For each $\xi < \mathfrak{c}$, let $e_\xi \in \mathbb{Z}^{(\mathfrak{c})}$ denote the vector supported at ξ and defined by

$$e_\xi(\eta) = \begin{cases} 1, & \text{if } \eta = \xi, \\ 0, & \text{if } \eta \neq \xi. \end{cases}$$

Lemma 2.1. *Let G be a torsion-free Abelian group of cardinality $\mathfrak{c}$. Then G may be identified with a subgroup of $\mathbb{Q}^{(\mathfrak{c})}$ in such a way that*

$$\mathbb{Z}^{(\mathfrak{c})} \leq G \leq \mathbb{Q}^{(\mathfrak{c})}.$$

In particular, for every $\xi < \mathfrak{c}$, the vector e_ξ belongs to G.

Proof. The canonical homomorphism

$$G \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} G, \qquad x \longmapsto 1 \otimes x,$$

is injective because G is torsion-free. Its image spans the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} G$. As $\mathbb{Q}$ is countable, the dimension of this vector space is $\mathfrak{c}$ because G has cardinality $\mathfrak{c}$.

Choose a basis $(1 \otimes b_\xi)_{\xi<\mathfrak{c}}$ from the spanning set given by the image of G, and identify $\mathbb{Q} \otimes_{\mathbb{Z}} G$ with $\mathbb{Q}^{(\mathfrak{c})}$ by sending $1 \otimes b_\xi$ to e_ξ. Under the resulting embedding of G, each b_ξ is identified with e_ξ, and hence $\mathbb{Z}^{(\mathfrak{c})} \leq G \leq \mathbb{Q}^{(\mathfrak{c})}$. □

For a finite tuple $z = (z_0, \ldots, z_{m-1})$ in an Abelian group put

$$\operatorname{Rel}(z) = \left\{ a \in \mathbb{Z}^m : \sum_{i<m} a_i z_i = 0 \right\}, \qquad \operatorname{ht}(a) = \max\bigl(\{0\} \cup \{|a_i| : i < m\}\bigr).$$

In a T_1 space, countable compactness is equivalent to the assertion that every countably infinite subset has an accumulation point [11, Theorem 3.10.3(v)]. Also, if $u : \mathbb{N} \to X$ is injective, p is free, and $p\text{-}\lim_n u(n) = x$, then x is an accumulation point of $u[\mathbb{N}]$.

The following observation will be applied to show that the group topologies we construct are countably compact. This is a standard result and we include a proof for completeness.

Lemma 2.2. *Let G be a Hausdorff topological group. Suppose that for every injective $s : \mathbb{N} \to G$ there exist a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$, a free ultrafilter p on $\mathbb{N}$, and a point $b \neq 0$ such that*

$$p\text{-}\lim_n s(\varphi(n)) = b. \tag{1}$$

Then G is countably compact and every convergent sequence in G is eventually constant.

Proof. Given a countably infinite set $A \subseteq G$, enumerate it injectively by s and apply the hypothesis. It follows that A has an accumulation point, so G is countably compact.

Now suppose that an injective sequence s converges to x. Define $t : \mathbb{N} \to G$ by $t(n) = s(n) - x$. The sequence t is injective and converges to 0. Apply the hypothesis to t, obtaining φ, p and $b \neq 0$. Since φ is strictly increasing, $t \circ \varphi$ still converges to 0, and since p is free, it follows that $p\text{-}\lim_n t(\varphi(n)) = 0$. This contradicts uniqueness of limits in a Hausdorff space, so s cannot be injective.

Finally let s be any sequence converging to x. If its range is finite, then $s(n) = x$ eventually. If its range is infinite, choose recursively a strictly increasing subsequence with pairwise distinct values. That subsequence still converges to x, contradicting the preceding paragraph. □

Suppose now that $(\psi_j)_{j\in J}$ is a family of homomorphisms in $\mathrm{Hom}(G, \mathbb{T})$ that separates points: whenever $x \neq y$, there exists $j \in J$ such that $\psi_j(x) \neq \psi_j(y)$. Define

$$\Delta : G \longrightarrow \mathbb{T}^J, \qquad \Delta(x) = (\psi_j(x))_{j\in J}.$$

Give G the coarsest topology that makes every ψ_j continuous, that is, the initial topology induced by the family $(\psi_j)_{j\in J}$. This is a Hausdorff group topology, and Δ is a topological embedding onto its image. Thus, once these homomorphisms have been constructed, only (1) remains to obtain all the topological conclusions.

3. Uniform finite approximation by circle-valued homomorphisms

A basic ingredient in the construction is the following finite interpolation problem. Given elements $z_0, \dots, z_{m-1}$ of an Abelian group and prescribed target values $t_0, \dots, t_{m-1} \in \mathbb{T}$, we seek a single homomorphism from the ambient group to $\mathbb{T}$ whose value at each z_i is "close" to t_i.

The following Kronecker-type lemma shows that, for approximate realization to a fixed accuracy, it is enough to check compatibility only with the relations whose coefficients are bounded by a suitable finite constant.

Lemma 3.1 (Uniform finite Kronecker lemma)**.** *For every integer $m \geq 1$ and every $\varepsilon > 0$ there exists $q \in \mathbb{N}$ with the following property.*

For every Abelian group G, for every $z = (z_i)_{i<m} \in G^m$, and for every $t = (t_i)_{i<m} \in \mathbb{T}^m$, if for every relation $a = (a_i)_{i<m} \in \mathrm{Rel}(z)$ with $\mathrm{ht}(a) \leq q$ we have $\sum_{i<m} a_i t_i = 0$, then there exists $\psi \in \mathrm{Hom}(G, \mathbb{T})$ such that

$$\max_{i<m} \|\psi(z_i) - t_i\| < \varepsilon.$$

We note that the integer q is independent of G, z, and t.

We dedicate this section to proving Lemma 3.1. First, we isolate the analytic ingredients that will be needed. For each integer $m \geq 1$, on $\mathbb{T}^m$ we use the maximum metric

$$\rho_\infty(u, v) = \max_{i<m} \|u_i - v_i\|.$$

Let $\mathbb{S}^1 = \{w \in \mathbb{C} : |w| = 1\}$. The homomorphism $r \mapsto e^{2\pi i r}$ from $\mathbb{R}$ onto $\mathbb{S}^1$ has kernel $\mathbb{Z}$, so it induces the continuous group isomorphism

$$E : \mathbb{T} = \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^1, \qquad E(r + \mathbb{Z}) = e^{2\pi i r}.$$

For $a = (a_i)_{i<m} \in \mathbb{Z}^m$, define the continuous homomorphism

$$\phi_a : \mathbb{T}^m \longrightarrow \mathbb{S}^1, \qquad \phi_a(u) = E\left(\sum_{i<m} a_i u_i\right).$$

The only compactness result for families of functions that we need is the following compact-metric form of Arzelà–Ascoli. For a proof, see, e.g., [11, Theorem 8.2.10].

Lemma 3.2 (Arzelà–Ascoli)**.** *Let X be a compact metric space. If $\mathcal{F} \subseteq C(X, \mathbb{R})$ is equicontinuous and uniformly bounded, then its closure in the uniform norm is compact.*

For the following standard complex form of the Stone–Weierstrass theorem, see [26, Theorem 7.33]. Recall that a subset $\mathcal{A} \subseteq C(X, \mathbb{C})$ is uniformly dense in $C(X, \mathbb{C})$ if it is dense in the topology induced by the uniform norm $\|\cdot\|_\infty$.

Theorem 3.3 (Complex Stone–Weierstrass theorem)**.** *Let X be a compact Hausdorff space, and let $\mathcal{A} \subseteq C(X, \mathbb{C})$ be a complex subalgebra. If $\mathcal{A}$ contains the constant functions, separates the points of X, and is closed under complex conjugation, then $\mathcal{A}$ is uniformly dense in $C(X, \mathbb{C})$.*

Lemma 3.4. *For every integer $m \geq 1$, the complex linear span of $\{\phi_a : a \in \mathbb{Z}^m\}$ is uniformly dense in $C(\mathbb{T}^m, \mathbb{C})$.*

Proof. Their complex linear span is a unital algebra because $\phi_0 = 1$ and $\phi_a \phi_b = \phi_{a+b}$, and it is closed under complex conjugation because $\overline{\phi_a} = \phi_{-a}$. It also separates points: if $u \neq v$, then $u_j \neq v_j$ for some $j < m$, and the homomorphism ϕ_{e_j}, where e_j is the jth standard basis vector, distinguishes u and v. Theorem 3.3 now applies. □

We will call each element of the span of $\{\phi_a : a \in \mathbb{Z}^m\}$ a *trigonometric polynomial* on $\mathbb{T}^m$.

We now combine these results to obtain the lemma below. In its statement, 1-Lipschitz means Lipschitz with constant 1 with respect to the metric ρ_∞ on $\mathbb{T}^m$ and the usual metric on $\mathbb{R}$.

Lemma 3.5. *For every integer $m \geq 1$ and every $\eta > 0$, there exists a finite set $S \subseteq \mathbb{Z}^m$ such that for every 1-Lipschitz function $f : \mathbb{T}^m \to [0, 1/2]$ there exist complex coefficients $(c_a)_{a \in S}$ satisfying*

$$\left\| f - \sum_{a \in S} c_a \phi_a \right\|_\infty < \eta.$$

Proof. Let $\mathcal{L}_m$ be the set of all such functions, regarded as a subset of $C(\mathbb{T}^m, \mathbb{R})$. This family is uniformly bounded, equicontinuous (as all its members are 1-Lipschitz), and closed under uniform limits. Hence it is closed in $C(\mathbb{T}^m, \mathbb{R})$ and compact by Lemma 3.2.

For each $f \in \mathcal{L}_m$, use Lemma 3.4 to choose a trigonometric polynomial P_f such that $\|f - P_f\|_\infty < \eta/2$. The uniform balls of radius $\eta/2$ centered at the members of $\mathcal{L}_m$ cover $\mathcal{L}_m$; choose a finite subcover with centers $f_0, \ldots, f_N$. For each $j \le N$, choose a finite set $S_j \subseteq \mathbb{Z}^m$ such that $P_{f_j} \in \operatorname{span}\{\phi_a : a \in S_j\}$, and set $S = \bigcup_{j \le N} S_j$.

If f belongs to the ball centered at f_j, then

$$\|f - P_{f_j}\|_\infty \le \|f - f_j\|_\infty + \|f_j - P_{f_j}\|_\infty < \eta.$$

This proves the assertion. $\square$

For an integer $m \ge 1$ and a subgroup $R \le \mathbb{Z}^m$, put

$$H_R = \left\{ u \in \mathbb{T}^m : \sum_{i<m} a_i u_i = 0 \text{ for every } a \in R \right\}.$$

Recall that if H is a compact Hausdorff Abelian topological group, its *normalized Haar measure* is the unique regular Borel measure μ on H such that $\mu(H) = 1$ and $\mu(x + B) = \mu(B)$ for every $x \in H$ and every Borel set $B \subseteq H$. Thus μ is a probability measure invariant under translations; in particular, for every continuous $g : H \to \mathbb{C}$ and every $x \in H$,

$$\int_H g(x + h)\, d\mu(h) = \int_H g(h)\, d\mu(h).$$

Existence and uniqueness are standard; see [19, Chapter VII, §§ 2–3].

Lemma 3.6. *For every integer $m \ge 1$ and every subgroup $R \le \mathbb{Z}^m$, the set H_R is a compact subgroup of $\mathbb{T}^m$, and*

$$\{a \in \mathbb{Z}^m : \phi_a(h) = 1 \text{ for every } h \in H_R\} = R. \tag{2}$$

If μ_R is normalized Haar measure on H_R, then, for every $a \in \mathbb{Z}^m$ and $s \in \mathbb{T}^m$,

$$\int_{H_R} \phi_a(s + h)\, d\mu_R(h) = \begin{cases} \phi_a(s), & a \in R, \\ 0, & a \notin R. \end{cases} \tag{3}$$

Proof. The set H_R is an intersection of kernels of continuous homomorphisms from $\mathbb{T}^m$ to $\mathbb{T}$, so it is a closed subgroup of the compact group $\mathbb{T}^m$. The inclusion from right to left in (2) follows from the definition of H_R.

For the reverse inclusion, let $\pi : \mathbb{Z}^m \to \mathbb{Z}^m/R$ be the quotient map and suppose that $a \notin R$. Since $\pi(a) \ne 0$, there is a homomorphism $\phi : \mathbb{Z}^m/R \to \mathbb{T}$ such that $\phi(\pi(a)) \ne 0$: first define it on $\langle \pi(a) \rangle$ and then use the injectivity of $\mathbb{T}$ to extend it. For $i < m$, put $u_i = \phi(\pi(e_i))$. If $r = (r_i)_{i<m} \in R$, then

$$\sum_{i<m} r_i u_i = \phi\left(\pi\left(\sum_{i<m} r_i e_i\right)\right) = \phi(\pi(r)) = 0,$$

so $u = (u_i)_{i<m} \in H_R$. On the other hand,

$$\sum_{i<m} a_i u_i = \phi(\pi(a)) \neq 0.$$

Thus $\phi_a(u) \neq 1$, proving (2).

Put

$$I_a = \int_{H_R} \phi_a(h)\, d\mu_R(h).$$

If $a \in R$, then ϕ_a is identically 1 on H_R, so $I_a = 1$. If $a \notin R$, identity (2) gives $h_0 \in H_R$ with $\phi_a(h_0) \neq 1$. Translation invariance gives

$$I_a = \int_{H_R} \phi_a(h_0 + h)\, d\mu_R(h) = \int_{H_R} \phi_a(h_0)\phi_a(h)\, d\mu_R(h) = \phi_a(h_0) I_a,$$

and hence $I_a = 0$. Finally, $\phi_a(s+h) = \phi_a(s)\phi_a(h)$, so

$$\int_{H_R} \phi_a(s+h)\, d\mu_R(h) = \phi_a(s) I_a,$$

which proves (3). □

Now we are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Set $\eta = \varepsilon/4$, and let $S \subseteq \mathbb{Z}^m$ be supplied by Lemma 3.5. Put

$$q = \max(\{1\} \cup \{\mathrm{ht}(a) : a \in S\}).$$

Claim. *For every subgroup $R \leq \mathbb{Z}^m$ and every $t \in \mathbb{T}^m$, if*

$$\sum_{i<m} a_i t_i = 0 \qquad \text{for every } a \in R \cap [-q, q]^m,$$

then $\rho_\infty(t, H_R) < \varepsilon$.

Proof of the claim. Fix a subgroup $R \leq \mathbb{Z}^m$ and $t \in \mathbb{T}^m$ such that

$$\sum_{i<m} a_i t_i = 0 \text{ for every } a \in R \cap [-q, q]^m. \tag{4}$$

Define $d : \mathbb{T}^m \to \mathbb{R}$ by

$$d(u) = \rho_\infty(u, H_R) = \inf_{h \in H_R} \rho_\infty(u, h).$$

The function d is 1-Lipschitz. Moreover, $0 \in H_R$, and hence

$$0 \leq d(u) \leq \rho_\infty(u, 0) \leq \frac{1}{2} \qquad (u \in \mathbb{T}^m).$$

Lemma 3.5, applied to d, gives coefficients $(c_a)_{a \in S} \in \mathbb{C}^S$ such that the trigonometric polynomial

$$P(u) := \sum_{a \in S} c_a \phi_a(u)$$

satisfies $\|P - d\|_\infty < \eta$.

For $s \in \mathbb{T}^m$, set

$$A(s) = \int_{H_R} P(s+h)\, d\mu_R(h).$$

Equation (3) yields

$$A(s) = \sum_{a\in S\cap R} c_a\phi_a(s). \tag{5}$$

By the definition of q, $S\cap R \subseteq R\cap[-q,q]^m$. Thus (4) implies $\phi_a(t)=1$ for every $a\in S\cap R$, and (5) gives

$$A(t) = \sum_{a\in S\cap R} c_a\phi_a(t) = \sum_{a\in S\cap R} c_a = \sum_{a\in S\cap R} c_a\phi_a(0) = A(0). \tag{6}$$

Because d vanishes on H_R and μ_R is a probability measure,

$$|A(0)| = \left|\int_{H_R} (P(h)-d(h))\, d\mu_R(h)\right| \le \|P-d\|_\infty < \eta. \tag{7}$$

Since H_R is a subgroup and ρ_∞ is translation invariant, $d(t+h)=d(t)$ for every $h\in H_R$. Consequently, $d(t)=\int_{H_R} d(t+h)\,d\mu_R(h)$, and therefore

$$|d(t)-A(t)| = \left|\int_{H_R} (d(t+h)-P(t+h))\, d\mu_R(h)\right| < \eta. \tag{8}$$

Combining (6)–(8), we obtain

$$d(t) = |d(t)| \le |d(t)-A(t)| + |A(0)| < 2\eta < \varepsilon.$$

□

Now let G be an Abelian group, let $z=(z_i)_{i<m}\in G^m$ and $t=(t_i)_{i<m}\in\mathbb{T}^m$ satisfy the hypothesis of the lemma, and take $R=\mathrm{Rel}(z)$. Then t satisfies the hypothesis of the claim for this R. Hence $\rho_\infty(t,H_R)<\varepsilon$, so there exists $u\in H_R$ such that $\rho_\infty(u,t)<\varepsilon$. Define a map on the subgroup $L=\langle z_i : i<m\rangle \le G$ by

$$\varphi\left(\sum_{i<m} n_i z_i\right) = \sum_{i<m} n_i u_i.$$

To see that this is well defined, suppose that $n,n'\in\mathbb{Z}^m$ satisfy $\sum_{i<m} n_i z_i = \sum_{i<m} n_i' z_i$. Then $n-n'\in\mathrm{Rel}(z)=R$, and since $u\in H_R$,

$$\sum_{i<m} n_i u_i - \sum_{i<m} n_i' u_i = \sum_{i<m} (n_i-n_i')u_i = 0.$$

Clearly, φ is a homomorphism from L to $\mathbb{T}$. By injectivity of $\mathbb{T}$, the map φ extends to some $\psi\in\mathrm{Hom}(G,\mathbb{T})$. For each $i<m$ we have $\psi(z_i)=u_i$, and therefore

$$\max_{i<m} \|\psi(z_i)-t_i\| = \rho_\infty(u,t) < \varepsilon.$$

□

4. Bounded integer relations and deletion

The goal of this section is Lemma 4.5. Given a homomorphism $\psi : G \to \mathbb{T}$ and finite sets $A, X \subseteq G$, that lemma produces a large subset $Y \subseteq X$ and a homomorphism $\psi' : G \to \mathbb{T}$ which approximately preserves the values of ψ on A and is small on Y. Lemma 3.1 reduces this construction to eliminating bounded-height integer relations that mix elements of A with elements of Y. We now develop the finite combinatorial deletion argument needed to achieve this.

Definition 4.1. Let G be an Abelian group. For a finite set $X \subseteq G$, put

$$\mathrm{Rel}_G(X) = \left\{ c = (c_x)_{x\in X} \in \mathbb{Z}^X : \sum_{x\in X} c_x\, x = 0 \right\}.$$

This is the set-indexed version of the notation $\mathrm{Rel}(z)$ introduced in Section 2. For $c \in \mathbb{Z}^X$, put

$$\mathrm{ht}(c) = \max\bigl(\{0\} \cup \{|c_x| : x \in X\}\bigr).$$

For $M \in \mathbb{N}$, the set X is *M-independent* if the zero vector is the only $c \in \mathrm{Rel}_G(X)$ satisfying $\mathrm{ht}(c) \le M$.

For finite sets $A, Y \subseteq G$, put

$$\mathrm{Rel}_G(A, Y) = \left\{ (b, c) \in \mathbb{Z}^A \times \mathbb{Z}^Y : \sum_{a\in A} b_a\, a + \sum_{y\in Y} c_y\, y = 0 \right\}.$$

An element $(b, c) \in \mathrm{Rel}_G(A, Y)$ is *mixed* if $b \ne 0$ and $c \ne 0$. We set $\mathrm{ht}(b, c) = \max\{\mathrm{ht}(b), \mathrm{ht}(c)\}$.

Lemma 4.2. *Let G be a torsion-free Abelian group and $S \subseteq G$ be infinite. Given $M, N \in \mathbb{N}$, there exists an M-independent subset of S of cardinality N.*

Proof. We construct, by induction on $k \le N$, an M-independent set $B_k \subseteq S$ such that $|B_k| = k$.

For $k = 0$, take $B_0 = \varnothing$, which is trivially M-independent. Suppose that $k < N$ and that

$$B_k = \{x_i : i < k\} \subseteq S$$

has already been chosen and is M-independent. We shall choose $x \in S \setminus B_k$ such that $B_k \cup \{x\}$ remains M-independent.

Indeed, given $x \in S \setminus B_k$, if $B_k \cup \{x\}$ is not M-independent, then there exist integers q and $(c_i)_{i<k}$, not all zero, with

$$|q| \le M, \qquad |c_i| \le M \quad (i < k),$$

such that

$$qx + \sum_{i<k} c_i x_i = 0.$$

We claim that $q \ne 0$. Indeed, if $q = 0$, then the family $c \in \mathbb{Z}^{B_k}$ defined by $c_{x_i} = c_i$ for $i < k$ would be a nonzero element of $\mathrm{Rel}_G(B_k)$ with $\mathrm{ht}(c) \le M$, contradicting the M-independence of B_k. Hence every bad choice of x satisfies an equation of the form

$$qx = -\sum_{i<k} c_i x_i, \qquad 0 < |q| \le M, \quad |c_i| \le M \quad (i < k).$$

There exist only finitely many possible choices of q and $(c_i)_{i<k}$, and each such choice determines at most one possible value of x. In fact, if both x and x' satisfy the same equation, then

$$q(x - x') = 0.$$

Since G is torsion-free and $q \neq 0$, it follows that $x = x'$. Consequently, only finitely many elements of S are forbidden.

Because S is infinite and B_k is finite, we may choose $x_k \in S \setminus B_k$ outside this finite set of forbidden elements. Then $B_{k+1} = B_k \cup \{x_k\}$ is M-independent and has cardinality $k + 1$. This completes the induction step and the proof. □

Lemma 4.3. *For $R, Q \in \mathbb{N}$ there is $B = B(R, Q) \in \mathbb{N}$ such that, whenever $s \in \mathbb{N}$, $s \le R$, and $\mathbf{v} = (v_0, \ldots, v_s) \in (\mathbb{Z}^s)^{s+1}$ has all coordinates of absolute value at most Q, there exists $\lambda \in \mathrm{Rel}(\mathbf{v}) \setminus \{0\}$ such that $\mathrm{ht}(\lambda) \le B$.*

Proof. Fix $s \le R$ and let $\mathbf{v} = (v_0, \ldots, v_s)$ be a tuple satisfying the hypotheses. Consider the linear map $T_{\mathbf{v}} : \mathbb{Q}^{s+1} \longrightarrow \mathbb{Q}^s$ defined by

$$T_{\mathbf{v}}(c_0, \ldots, c_s) = \sum_{i \le s} c_i v_i.$$

As $T_{\mathbf{v}}$ cannot be injective, there is a nonzero tuple $(c_0, \ldots, c_s) \in \mathbb{Q}^{s+1}$ such that

$$\sum_{i \le s} c_i v_i = 0.$$

Choose a positive integer d which is a common multiple of the denominators of $c_0, \ldots, c_s$, and put

$$\lambda_i = d c_i \qquad (i \le s).$$

Then each λ_i is an integer, the tuple $(\lambda_0, \ldots, \lambda_s)$ is nonzero, and

$$\sum_{i \le s} \lambda_i v_i = d \sum_{i \le s} c_i v_i = 0.$$

Thus $\mathrm{Rel}(\mathbf{v}) \setminus \{0\}$ is nonempty for every tuple $\mathbf{v}$ satisfying the hypotheses. Let

$$b(\mathbf{v}) = \min \{\mathrm{ht}(\lambda) : \lambda \in \mathrm{Rel}(\mathbf{v}) \setminus \{0\}\}$$

and

$$\mathcal{V}_s = \left\{ (v_0, \ldots, v_s) \in (\mathbb{Z}^s)^{s+1} : |v_i(j)| \le Q \text{ for every } i \le s \text{ and } j < s \right\}.$$

Both $\mathcal{V}_s$ and the set

$$\mathcal{V} = \bigcup_{s \le R} (\{s\} \times \mathcal{V}_s)$$

are finite. We may therefore define

$$B(R,Q) = \max\left(\{1\} \cup \{b(\mathbf{v}) : s \leq R \text{ and } \mathbf{v} \in \mathcal{V}_s\}\right),$$

which is a well-defined positive integer depending only on R and Q. By the definition of $b(\mathbf{v})$, there is a nonzero $\lambda = (\lambda_0, \dots, \lambda_s) \in \mathrm{Rel}(\mathbf{v})$ such that

$$\mathrm{ht}(\lambda) = b(\mathbf{v}) \leq B(R,Q).$$

This is the required tuple. ☐

Lemma 4.4 (Bounded deletion)**.** *For every $R, Q \in \mathbb{N}$ there exists $M = M(R,Q) \in \mathbb{N}$ with the following property. If G is a torsion-free Abelian group, $A, X \subseteq G$ are finite, $|A| \leq R$, and X is M-independent, then some $Y \subseteq X$ satisfies*

(i) $|X \setminus Y| \leq |A|$;
(ii) *there is no mixed* $(b,c) \in \mathrm{Rel}_G(A,Y)$ *such that* $\mathrm{ht}(b,c) \leq Q$.

Proof. Let $B = B(R,Q)$ be supplied by Lemma 4.3, and put

$$M = (R+1)BQ.$$

Choose a subset $Y \subseteq X$ of maximum cardinality which admits no mixed $(b,c) \in \mathrm{Rel}_G(A,Y)$ with $\mathrm{ht}(b,c) \leq Q$.

Write $s = |A|$ and suppose, towards a contradiction, that $X \setminus Y$ contains distinct elements $x_0, \dots, x_s$. By the maximality of $|Y|$, for each $i \leq s$ there is a mixed element

$$(\beta_i, \gamma_i) \in \mathrm{Rel}_G(A, Y \cup \{x_i\}) \qquad \text{with} \qquad \mathrm{ht}(\beta_i, \gamma_i) \leq Q.$$

Thus

$$\sum_{a \in A} \beta_i(a)\, a + \sum_{y \in Y} \gamma_i(y)\, y + \gamma_i(x_i)\, x_i = 0. \tag{9}$$

Moreover, $\gamma_i(x_i) \neq 0$, since otherwise $(\beta_i, \gamma_i|_Y)$ would be a mixed element of $\mathrm{Rel}_G(A,Y)$ contradicting the choice of Y.

Fix an arbitrary enumeration

$$A = \{a_0, \dots, a_{s-1}\}$$

and for each $i \leq s$, define

$$b_i = (\beta_i(a_0), \dots, \beta_i(a_{s-1})) \in \mathbb{Z}^s.$$

Apply Lemma 4.3 to $b_0, \dots, b_s \in \mathbb{Z}^s$. Since $s \leq R$, there are integers $\lambda_0, \dots, \lambda_s$, not all zero, such that

$$|\lambda_i| \leq B \quad (i \leq s), \qquad \sum_{i \leq s} \lambda_i b_i = 0.$$

Multiplying the i-th instance of (9) by λ_i and summing over $i \leq s$, the sum involving the elements of A vanishes, and we obtain

$$\sum_{y \in Y} \left(\sum_{i \leq s} \lambda_i \gamma_i(y) \right) y + \sum_{i \leq s} \lambda_i \gamma_i(x_i)\, x_i = 0.$$

Put $W = Y \cup \{x_0, \dots, x_s\}$ and define $e \in \mathbb{Z}^W$ by

$$e_y = \sum_{i \le s} \lambda_i \gamma_i(y) \quad (y \in Y), \qquad e_{x_i} = \lambda_i \gamma_i(x_i) \quad (i \le s).$$

The displayed equality says exactly that $e \in \mathrm{Rel}_G(W)$.

For each $y \in Y$, we have

$$|e_y| \le \sum_{i \le s} |\lambda_i| \, |\gamma_i(y)| \le (s+1)BQ \le M,$$

and for each $i \le s$,

$$|e_{x_i}| = |\lambda_i \gamma_i(x_i)| \le BQ \le M.$$

Moreover, $e \ne 0$: if $\lambda_{i_0} \ne 0$, then $e_{x_{i_0}} = \lambda_{i_0}\gamma_{i_0}(x_{i_0}) \ne 0$. Extending e by zero on $X \setminus W$ therefore gives a nonzero element of $\mathrm{Rel}_G(X)$ of height at most M. This contradicts the M-independence of X. Therefore $|X \setminus Y| \le |A|$, as required. □

Lemma 4.5. *Let G be a torsion-free Abelian group, let $A, X \subseteq G$ be finite, and let $\psi : G \to \mathbb{T}$ be a homomorphism. Fix $\varepsilon > 0$ and $R \in \mathbb{N}$ with $R \ge |A|$.*

For each positive integer $m \le R + |X|$, let $q(m, \varepsilon)$ be an integer supplied by Lemma 3.1, and let $Q \in \mathbb{N}$ satisfy

$$Q \ge \max\Big(\{1\} \cup \{q(m,\varepsilon) : 1 \le m \le R + |X|\}\Big).$$

Let $M = M(R, Q)$ be supplied by Lemma 4.4. If X is M-independent, then there exist a set $Y \subseteq X$ and a homomorphism $\psi' : G \to \mathbb{T}$ such that

(i) *$|X \setminus Y| \le |A|$;*
(ii) *there is no mixed $(b, c) \in \mathrm{Rel}_G(A, Y)$ such that $\mathrm{ht}(b, c) \le Q$;*
(iii) *$\|\psi'(a) - \psi(a)\| < \varepsilon$ for every $a \in A$;*
(iv) *$\|\psi'(y)\| < \varepsilon$ for every $y \in Y$.*

Proof. Suppose that X is M-independent. By Lemma 4.4, there exists a set $Y \subseteq X$ satisfying items (i) and (ii). Fix such a set Y.

We first observe that $A \cap Y = \varnothing$. Indeed, if $z \in A \cap Y$, the coefficient families $b \in \mathbb{Z}^A$ and $c \in \mathbb{Z}^Y$ defined by

$$b_z = 1, \quad c_z = -1, \qquad b_a = 0 \ (a \in A \setminus \{z\}), \qquad c_y = 0 \ (y \in Y \setminus \{z\})$$

would form a mixed element $(b, c) \in \mathrm{Rel}_G(A, Y)$ with $\mathrm{ht}(b, c) = 1 \le Q$, contradicting item (ii).

Since A and Y are disjoint, we may define a function $t : A \cup Y \to \mathbb{T}$ by

$$t(z) = \begin{cases} \psi(z), & \text{if } z \in A, \\ 0, & \text{if } z \in Y. \end{cases}$$

If $A \cup Y = \varnothing$, take $\psi' = \psi$. Items (iii) and (iv) are then vacuously satisfied. Suppose now that $A \cup Y \ne \varnothing$, and let $z = (z_i)_{i<m}$ enumerate this set without repetitions, so that

$$A \cup Y = \{z_i : i < m\},$$

for some $m \in \mathbb{N}$. Then, we have

$$m = |A| + |Y| \leq R + |X|,$$

and, by the choice of Q, $q(m, \varepsilon) \leq Q$.

Let $\mathbf{d} = (d_i)_{i<m} \in \mathrm{Rel}(z)$ satisfy $\mathrm{ht}(\mathbf{d}) \leq q(m, \varepsilon)$. We will prove that $\sum_{i<m} d_i t(z_i) = 0$. Since $\mathbf{d} \in \mathrm{Rel}(z)$, we have

$$\sum_{\substack{i<m \\ z_i \in A}} d_i z_i + \sum_{\substack{i<m \\ z_i \in Y}} d_i z_i = 0,$$

and, because the enumeration has no repetitions and $A \cap Y = \varnothing$, it determines $(b, c) \in \mathrm{Rel}_G(A, Y)$ by

$$b_{z_i} = d_i \quad (z_i \in A), \qquad c_{z_i} = d_i \quad (z_i \in Y).$$

$\mathrm{ht}(b, c) \leq q(m, \varepsilon) \leq Q$. By item (ii), the pair (b, c) is not mixed, so one of the following two alternatives holds:

$$d_i = 0 \quad \text{for every } i < m \text{ such that } z_i \in A,$$

or

$$d_i = 0 \quad \text{for every } i < m \text{ such that } z_i \in Y.$$

In the first case, all possibly nonzero coefficients correspond to elements of Y. Since $t(y) = 0$ for every $y \in Y$, we obtain

$$\sum_{i<m} d_i t(z_i) = 0.$$

In the second case, all possibly nonzero coefficients correspond to elements of A. Since ψ is a homomorphism, we obtain

$$\begin{aligned} \sum_{i<m} d_i t(z_i) &= \sum_{\substack{i<m \\ z_i \in A}} d_i \psi(z_i) \\ &= \psi\left(\sum_{\substack{i<m \\ z_i \in A}} d_i z_i\right) \\ &= \psi(0) = 0. \end{aligned}$$

Thus, in either case,

$$\sum_{i<m} d_i t(z_i) = 0.$$

Lemma 3.1 now provides a homomorphism $\psi' : G \to \mathbb{T}$ such that

$$\|\psi'(z_i) - t(z_i)\| < \varepsilon \qquad (i < m).$$

For every $a \in A$, we have $t(a) = \psi(a)$, and hence

$$\|\psi'(a) - \psi(a)\| < \varepsilon.$$

For every $y \in Y$, we have $t(y) = 0$, and hence $\|\psi'(y)\| < \varepsilon$. Thus items (iii) and (iv) also hold. $\square$

5. Block ultrafilters

Fix a torsion-free Abelian group G of cardinality $\mathfrak{c}$ and, using Lemma 2.1, identify it so that

$$\mathbb{Z}^{(\mathfrak{c})} \leq G \leq \mathbb{Q}^{(\mathfrak{c})}.$$

There are exactly $\mathfrak{c}$ injective sequences in G. Enumerate them without repetitions as $(s_\alpha)_{\alpha<\mathfrak{c}}$.

For each $\alpha < \mathfrak{c}$, the union of the supports of the terms of s_α is countable and hence bounded in $\mathfrak{c}$, because $\mathrm{cf}(\mathfrak{c}) > \omega$ by König's theorem. Recursively choose distinct coordinates $\iota(\alpha) < \mathfrak{c}$ so that ι is an injection and

$$\bigcup_{n\in\mathbb{N}} \operatorname{supp} s_\alpha(n) \subseteq \iota(\alpha) \qquad (\alpha < \mathfrak{c}).$$

We also fix the numerical parameters and the finite blocks used in the construction as follows.

- The sequence $(N_l)_{l\in\mathbb{N}}$ is defined recursively by

  $$N_l = (l+2)\left(2l+2+\sum_{j<l} N_j\right) \qquad (l \in \mathbb{N}),$$

  where the empty sum is understood to be equal to 0.

- For every $l \in \mathbb{N}$, put

$$S_l = \sum_{j<l} N_j, \quad R_l = S_l + 2l + 2 \quad \text{and} \quad \varepsilon_l = 2^{-(l+6)}. \tag{10}$$

- For every $l \in \mathbb{N}$ and every positive integer $m \leq R_l + N_l$, choose an integer $q(m, \varepsilon_l)$ supplied by Lemma 3.1. Then put

  $$Q_l = \max\Big(\{1\} \cup \{q(m,\varepsilon_l) : 1 \leq m \leq R_l + N_l\}\Big).$$

- For every $l \in \mathbb{N}$, let

  $$M_l = M(R_l, Q_l)$$

  be the integer supplied by Lemma 4.4 for the parameters R_l and Q_l.

- Finally, define

  $$I_l = \{S_l, S_l+1, \dots, S_l+N_l-1\} \qquad (l \in \mathbb{N}).$$

For every $l \in \mathbb{N}$, the definitions give

$$N_l = (l+2)R_l, \qquad \frac{R_l}{N_l} = \frac{1}{l+2} \quad \text{and} \quad \sum_{l\in\mathbb{N}} \varepsilon_l = \frac{1}{32}.$$

The intervals $(I_l)_{l\in\mathbb{N}}$ are pairwise disjoint and partition $\mathbb{N}$.

Lemma 5.1. *There exist strictly increasing maps $\varphi_\alpha : \mathbb{N} \to \mathbb{N}$ and sequences*

$$h_\alpha = s_\alpha \circ \varphi_\alpha : \mathbb{N} \longrightarrow G \qquad (\alpha < \mathfrak{c})$$

such that, for every $\alpha < \mathfrak{c}$, the following conditions hold:

- *for every $l \in \mathbb{N}$, the set*

$$X_{\alpha,l} = \{h_\alpha(n) - e_{\iota(\alpha)} : n \in I_l\}$$

has cardinality N_l and is M_l-independent;

- *for each $n \in \mathbb{N}$,*

$$\operatorname{supp} h_\alpha(n) \subseteq \iota(\alpha). \tag{11}$$

Proof. Fix $\alpha < \mathfrak{c}$, and define $\delta_\alpha : \mathbb{N} \to G$ by

$$\delta_\alpha(k) = s_\alpha(k) - e_{\iota(\alpha)} \qquad (k \in \mathbb{N}).$$

The map δ_α is injective.

Claim. *There exists a sequence $(K_{\alpha,l} : l \in \mathbb{N})$ of finite subsets of $\mathbb{N}$ satisfying the following conditions:*

(a) $|K_{\alpha,l}| = N_l$;
(b) $\max K_{\alpha,j} < \min K_{\alpha,l}$ *whenever* $j < l$;
(c) $\delta_\alpha[K_{\alpha,l}]$ *is* M_l*-independent.*

Proof of the claim. Suppose that $K_{\alpha,j}$ has been chosen for every $j < l$. Put

$$r_{\alpha,l} = \begin{cases} 0, & l = 0, \\ 1 + \max \bigcup_{j<l} K_{\alpha,j}, & l > 0. \end{cases}$$

Since the set

$$T_{\alpha,l} = \{\delta_\alpha(k) : k \geq r_{\alpha,l}\}$$

is infinite and G is torsion-free, Lemma 4.2 provides an M_l-independent set

$$Z_{\alpha,l} \subseteq T_{\alpha,l}$$

of cardinality N_l.

Let

$$K_{\alpha,l} = \delta_\alpha^{-1}[Z_{\alpha,l}].$$

As δ_α is injective and $Z_{\alpha,l} \subseteq T_{\alpha,l}$, we have $|K_{\alpha,l}| = |Z_{\alpha,l}| = N_l$, $K_{\alpha,l} \subseteq \mathbb{N} \setminus r_{\alpha,l}$, and $\delta_\alpha[K_{\alpha,l}] = Z_{\alpha,l}$. Therefore the three recursive requirements are satisfied. □

For every $l \in \mathbb{N}$, enumerate $K_{\alpha,l}$ increasingly as

$$K_{\alpha,l} = \{k_{\alpha,l,0} < k_{\alpha,l,1} < \cdots < k_{\alpha,l,N_l-1}\}.$$

We now define an auxiliary map $\varphi_\alpha : \mathbb{N} \to \mathbb{N}$ by

$$\varphi_\alpha(S_l + j) = k_{\alpha,l,j} \qquad (l \in \mathbb{N},\ j < N_l).$$

Since the intervals I_l partition $\mathbb{N}$, this defines φ_α on all of $\mathbb{N}$. The map φ_α is strictly increasing. Define

$$h_\alpha = s_\alpha \circ \varphi_\alpha : \mathbb{N} \to G.$$

For every $l \in \mathbb{N}$, we have

$$\begin{aligned} X_{\alpha,l} &= \{h_\alpha(n) - e_{\iota(\alpha)} : n \in I_l\} \\ &= \{s_\alpha(k_{\alpha,l,j}) - e_{\iota(\alpha)} : j < N_l\} \\ &= \delta_\alpha[K_{\alpha,l}] = Z_{\alpha,l}. \end{aligned}$$

Consequently, $X_{\alpha,l}$ has cardinality N_l and is M_l-independent.

Finally, since h_α is a subsequence of s_α,

$$\operatorname{supp} h_\alpha(n) \subseteq \iota(\alpha) \qquad (n \in \mathbb{N}).$$

□

Fix sequences $(h_\alpha)_{\alpha<\mathfrak{c}}$ as in Lemma 5.1 and a pairwise almost disjoint family

$$(A_\alpha)_{\alpha<\mathfrak{c}} \subseteq [\mathbb{N}]^\omega;$$

thus every A_α is infinite and $|A_\alpha \cap A_\beta| < \infty$ whenever $\alpha \neq \beta$. It is well known that such a family exists in ZFC. For every $\alpha < \mathfrak{c}$, define

$$\mathscr{D}_\alpha = \left\{ C \subseteq \mathbb{N} : \lim_{\substack{l \in A_\alpha \\ l \to \infty}} \frac{|I_l \setminus C|}{N_l} = 0 \right\}.$$

For every $\alpha < \mathfrak{c}$, the family $\mathscr{D}_\alpha$ is a free filter on $\mathbb{N}$. Indeed, it contains $\mathbb{N}$, does not contain the empty set, and is upward closed. If $C, C' \in \mathscr{D}_\alpha$, then

$$|I_l \setminus (C \cap C')| \leq |I_l \setminus C| + |I_l \setminus C'|,$$

so $C \cap C' \in \mathscr{D}_\alpha$. Finally, if $F \subseteq \mathbb{N}$ is finite, then $I_l \cap F = \varnothing$ for all sufficiently large l, and hence $\mathbb{N} \setminus F \in \mathscr{D}_\alpha$.

Lemma 5.2. *Let $\alpha < \mathfrak{c}$. Suppose that $B \subseteq A_\alpha$, $A_\alpha \setminus B$ is finite, and*

$$E_l \subseteq I_l, \qquad |E_l| \leq R_l \qquad (l \in B).$$

Then $\displaystyle\bigcup_{l \in B} (I_l \setminus E_l) \in \mathscr{D}_\alpha$.

Proof. Put $U = \displaystyle\bigcup_{l \in B} (I_l \setminus E_l)$.

Since $A_\alpha \setminus B$ is finite, there exists $l_0 \in \mathbb{N}$ such that $A_\alpha \cap [l_0, \infty) \subseteq B$. Suppose that $l \in A_\alpha$ and $l \geq l_0$. Then $l \in B$, and, as the intervals $(I_j)_{j\in\mathbb{N}}$ are pairwise disjoint, we have $I_l \cap U = I_l \setminus E_l$ and hence $I_l \setminus U = E_l$. Consequently,

$$\frac{|I_l \setminus U|}{N_l} = \frac{|E_l|}{N_l} \leq \frac{R_l}{N_l} = \frac{1}{l+2},$$

and thus

$$\lim_{\substack{l \in A_\alpha \\ l \to \infty}} \frac{|I_l \setminus U|}{N_l} = 0.$$

By the definition of $\mathscr{D}_\alpha$, it follows that $U \in \mathscr{D}_\alpha$. $\square$

For every $\alpha < \mathfrak{c}$, choose an ultrafilter p_α on $\mathbb{N}$ such that $\mathscr{D}_\alpha \subseteq p_\alpha$. The ultrafilter p_α is free because $\mathscr{D}_\alpha$ contains the cofinite filter. In particular, whenever $B \subseteq A_\alpha$, $A_\alpha \setminus B$ is finite, and

$$E_l \subseteq I_l, \qquad |E_l| \leq R_l \qquad (l \in B),$$

we have $\bigcup_{l \in B}(I_l \setminus E_l) \in p_\alpha$.

6. Local fusion of circle-valued homomorphisms

Fix $x \in G \setminus \{0\}$. We shall construct a homomorphism that does not vanish at x and satisfies the prescribed ultrafilter limits for all sequences whose distinguished coordinate belongs to a suitable countable set.

Lemma 6.1. *Let $(D_k)_{k\in\mathbb{N}}$ be a sequence defined recursively as follows:*

$$D_0 = \operatorname{supp} x;$$
$$D_{k+1} = D_k \cup \bigcup_{\substack{\alpha<\mathfrak{c} \\ \iota(\alpha)\in D_k}} \bigcup_{n\in\mathbb{N}} \operatorname{supp} h_\alpha(n).$$

Let $D = \bigcup_{k\in\mathbb{N}} D_k$.

Then D is countable. Moreover, for every $\alpha < \mathfrak{c}$ such that $\iota(\alpha) \in D$, we have

$$\operatorname{supp} h_\alpha(n) \subseteq D \qquad \text{for every } n \in \mathbb{N}.$$

Proof. A straightforward induction shows that D_k is countable for every $k \in \mathbb{N}$. Hence, D is countable.

Now suppose that $\iota(\alpha) \in D$. Then $\iota(\alpha) \in D_k$ for some k, and the definition of D_{k+1} gives $\operatorname{supp} h_\alpha(n) \subseteq D_{k+1} \subseteq D$ for every $n \in \mathbb{N}$. $\square$

From now on, put

$$G_D = G \cap \mathbb{Q}^{(D)} \qquad \text{and} \qquad \Gamma_D = \{\alpha < \mathfrak{c} : \iota(\alpha) \in D\}.$$

Choose an enumeration without repetitions $\Gamma_D = \{\alpha_j : j < J\}$, where $J \leq \omega$. Now, put

$$B_{\alpha_j} = A_{\alpha_j} \setminus \bigcup_{i<j} A_{\alpha_i} \qquad (j < J).$$

It is clear that $(B_{\alpha_j} : j < J)$ are pairwise disjoint and, for each $j < J$, we have

$$B_{\alpha_j} \subseteq A_{\alpha_j} \quad \text{and} \quad |A_{\alpha_j} \setminus B_{\alpha_j}| < \infty. \tag{12}$$

For each $l \in \mathbb{N}$ there is at most one $\alpha \in \Gamma_D$ such that $l \in B_\alpha$. We define

$$X_l = \begin{cases} X_{\alpha,l}, & \text{if } l \in B_\alpha \text{ for some } \alpha \in \Gamma_D, \\ \varnothing, & \text{if } l \notin \bigcup\limits_{\alpha\in\Gamma_D} B_\alpha. \end{cases} \tag{13}$$

Lemma 6.2. *For every $l \in \mathbb{N}$,*

$$X_l \subseteq G_D, \qquad |X_l| \leq N_l, \qquad X_l \text{ is } M_l\text{-independent.} \tag{14}$$

Proof. If $X_l = \varnothing$, all three assertions are immediate. Otherwise, $X_l = X_{\alpha,l}$ for the unique $\alpha \in \Gamma_D$ such that $l \in B_\alpha$. Since $\iota(\alpha) \in D$, Lemma 6.1 shows that

$$X_{\alpha,l} = \{h_\alpha(n) - e_{\iota(\alpha)} : n \in I_l\} \subseteq G_D.$$

Then Lemma 5.1 shows that $|X_l| = |X_{\alpha,l}| = N_l$ and that $X_l = X_{\alpha,l}$ is M_l-independent. ☐

Since D is countable, so is $\mathbb{Q}^{(D)}$, and hence G_D is countable. Fix a surjective sequence $(g_l)_{l\in\mathbb{N}}$ in G_D, repeating elements if needed.

Lemma 6.3. *There is a homomorphism $\psi_0 : G_D \to \mathbb{T}$ such that*

$$\psi_0(x) = \frac{1}{2} + \mathbb{Z}. \tag{15}$$

Proof. Since G_D is torsion-free and $x \neq 0$, the cyclic group $\langle x\rangle$ is isomorphic to $\mathbb{Z}$. The assignment

$$nx \longmapsto \frac{n}{2} + \mathbb{Z}$$

defines a homomorphism from $\langle x\rangle$ to $\mathbb{T}$. Since $\mathbb{T}$ is an injective Abelian group, this homomorphism extends to G_D. ☐

Proposition 6.4. *There exists a homomorphism $\psi_D : G_D \to \mathbb{T}$ such that*

(i) $\psi_D(x) \neq 0$;
(ii) *for every* $\alpha \in \Gamma_D$,

$$p_\alpha\text{-}\lim_n \psi_D(h_\alpha(n)) = \psi_D(e_{\iota(\alpha)}). \tag{16}$$

Proof. Let ψ_0 be supplied by Lemma 6.3. We recursively construct sequences $(\psi_l)_{l\in\mathbb{N}}$, $(A_l)_{l\in\mathbb{N}}$, and $(Y_l)_{l\in\mathbb{N}}$ such that, for each $l \in \mathbb{N}$:

(i) $\psi_l : G_D \to \mathbb{T}$ is a homomorphism;
(ii) $A_l = \bigcup_{j<l} Y_j \cup \{x\} \cup \{g_0, \dots, g_l\}$;
(iii) $Y_l \subseteq X_l$;
(iv) $|X_l \setminus Y_l| \le R_l$;
(v) $\|\psi_{l+1}(a) - \psi_l(a)\| < \varepsilon_l$ for every $a \in A_l$;
(vi) $\|\psi_{l+1}(y)\| < \varepsilon_l$ for every $y \in Y_l$.

Having defined $(\psi_i)_{i\le l}$, $(A_i)_{i\le l}$, and $(Y_i)_{i<l}$, we show how to define A_l, Y_l, and ψ_{l+1}.

Start by defining A_l as in item (ii). At each preceding step $j < l$, we have $Y_j \subseteq X_j$; hence, by (14), $|Y_j| \le |X_j| \le N_j$. It follows from (10) that

$$\begin{aligned}|A_l| &\le \sum_{j<l} |Y_j| + 1 + (l+1)\\ &\le \sum_{j<l} N_j + l + 2\\ &= S_l + l + 2 \le R_l.\end{aligned}$$

We now apply Lemma 4.5 with $G = G_D$, $A = A_l$, $X = X_l$, $\psi = \psi_l$, $\varepsilon = \varepsilon_l$, $R = R_l$, and $Q = Q_l$. In the notation of that lemma, $M = M(R_l, Q_l) = M_l$.

This application is valid because $|A_l| \le R_l$, X_l is M_l-independent, and, since $|X_l| \le N_l$, every positive integer $m \le R_l + |X_l|$ satisfies $m \le R_l + N_l$, so $q(m, \varepsilon_l) \le Q_l$ by the definition of Q_l. We obtain a set $Y_l \subseteq X_l$ and a homomorphism $\psi_{l+1} : G_D \to \mathbb{T}$ satisfying items (iii)–(vi). This completes the recursive construction.

We next prove that the sequence $(\psi_l(g))_{l\in\mathbb{N}}$ converges for every $g \in G_D$. Choose m such that $g_m = g$. For every $l \ge m$ we have $g \in A_l$, and therefore

$$\|\psi_{l+1}(g) - \psi_l(g)\| < \varepsilon_l.$$

Since $\sum_l \varepsilon_l < \infty$, the sequence $(\psi_l(g))_l$ is Cauchy in $\mathbb{T}$. The circle group is a complete metric space, so a limit exists. Define

$$\psi_D(g) = \lim_{l\to\infty} \psi_l(g) \qquad (g \in G_D).$$

As addition in $\mathbb{T}$ is continuous and ψ_D is the pointwise limit of homomorphisms, ψ_D is a homomorphism.

Since $x \in A_l$ for every l, (15) and item (v) give

$$\left\|\psi_D(x) - \left(\frac{1}{2} + \mathbb{Z}\right)\right\| \le \sum_{l\in\mathbb{N}} \|\psi_{l+1}(x) - \psi_l(x)\| \le \sum_{l\in\mathbb{N}} \varepsilon_l = \frac{1}{32} < \frac{1}{2}.$$

In particular, $\psi_D(x) \neq 0$.

Claim. *For all $l \in \mathbb{N}$ and all $y \in Y_l$, we have*

$$\|\psi_D(y)\| < 2\varepsilon_l. \tag{17}$$

Proof of the claim. Fix $l \in \mathbb{N}$ and $y \in Y_l$. For every $k > l$, item (ii) gives $y \in A_k$, so item (v) gives $\|\psi_{k+1}(y) - \psi_k(y)\| < \varepsilon_k$. Therefore,

$$\begin{aligned}\|\psi_D(y)\| &\le \|\psi_{l+1}(y)\| + \sum_{k>l} \|\psi_{k+1}(y) - \psi_k(y)\| \\ &< \varepsilon_l + \sum_{k>l} \varepsilon_k = 2\varepsilon_l.\end{aligned}$$

□

Now we verify (16). Fix $\alpha \in \Gamma_D$. If $l \in B_\alpha$, then the definition (13) gives $X_l = X_{\alpha,l}$. Define

$$E_l = \{n \in I_l : h_\alpha(n) - e_{\iota(\alpha)} \notin Y_l\} \qquad (l \in B_\alpha).$$

The map $n \longmapsto h_\alpha(n) - e_{\iota(\alpha)}$ is a bijection from I_l onto $X_{\alpha,l} = X_l$. Consequently, by item (iv),

$$|E_l| = |X_l \setminus Y_l| \le R_l \qquad (l \in B_\alpha).$$

Equation (12) and Lemma 5.2 now give $U_\alpha = \bigcup_{l\in B_\alpha} (I_l \setminus E_l) \in p_\alpha$.

We now intend to show that

$$p_\alpha\text{-}\lim_n \psi_D(h_\alpha(n) - e_{\iota(\alpha)}) = 0. \tag{18}$$

Let $n \in U_\alpha$, and let l be the unique index such that $n \in I_l$. Then $l \in B_\alpha$, $n \notin E_l$, and hence $h_\alpha(n) - e_{\iota(\alpha)} \in Y_l$. It follows from (17) that

$$\|\psi_D(h_\alpha(n) - e_{\iota(\alpha)})\| \le 2\varepsilon_l. \tag{19}$$

Fix $\delta > 0$ and choose $L \in \mathbb{N}$ such that $2\varepsilon_L < \delta$. The set $F_L = \bigcup_{l<L} I_l$ is finite, so, since p_α is free,

$$V_L = U_\alpha \cap (\mathbb{N} \setminus F_L) \in p_\alpha.$$

If $n \in V_L$ and $n \in I_l$, then $n \notin F_L$ implies $l \ge L$. Since $(\varepsilon_l)_l$ is decreasing, (19) gives

$$\|\psi_D(h_\alpha(n) - e_{\iota(\alpha)})\| \le 2\varepsilon_l \le 2\varepsilon_L < \delta.$$

Since $\delta > 0$ was arbitrary, (18) follows.

Finally, the additivity of ψ_D gives

$$\psi_D(h_\alpha(n)) = \psi_D(h_\alpha(n) - e_{\iota(\alpha)}) + \psi_D(e_{\iota(\alpha)}),$$

and taking the p_α-limit proves (16). □

The argument uses only that the local group G_D is countable, torsion-free, and Abelian.

7. Extending the homomorphisms

The previous section solves the problem on a countable subgroup: for each nonzero $x \in G$, it constructs a homomorphism on G_D that does not vanish at x and satisfies every prescribed limit equation associated with a coordinate in D, where D is a suitable countable subset of $\mathfrak{c}$. We now extend this homomorphism to all of G.

Proposition 7.1. *Let $D \subseteq \mathfrak{c}$ be such that, for every $\alpha < \mathfrak{c}$,*

$$\iota(\alpha) \in D \quad \Longrightarrow \quad \operatorname{supp} h_\alpha(n) \subseteq D \quad \text{for every } n \in \mathbb{N}.$$

Suppose that $\psi_D : G_D \to \mathbb{T}$ satisfies (16) whenever $\iota(\alpha) \in D$. Then ψ_D extends to a homomorphism $\psi : G \to \mathbb{T}$ satisfying

$$p_\alpha\text{-}\lim_n \psi(h_\alpha(n)) = \psi(e_{\iota(\alpha)}) \qquad (\alpha < \mathfrak{c}).$$

Proof. Since $\mathbb{T}$ is an injective Abelian group, ψ_D extends from the subgroup G_D of $\mathbb{Q}^{(D)}$ to a homomorphism

$$\widetilde{\psi}_D : \mathbb{Q}^{(D)} \longrightarrow \mathbb{T}.$$

By transfinite recursion, we construct homomorphisms $\theta_\xi : \mathbb{Q} \to \mathbb{T}$ for each $\xi < \mathfrak{c}$. If $\xi \in D$, let

$$\theta_\xi(q) = \widetilde{\psi}_D(q e_\xi) \qquad (q \in \mathbb{Q}).$$

Suppose next that $\xi \notin D$ and $\xi \in \iota[\mathfrak{c}]$. Since ι is injective, there is a unique $\alpha < \mathfrak{c}$ such that $\xi = \iota(\alpha)$. By (11), all coordinates in $h_\alpha(n)$ precede ξ, so the values

$$w_{\alpha,n} = \sum_{\eta \in \operatorname{supp} h_\alpha(n)} \theta_\eta(h_\alpha(n)(\eta)) \tag{20}$$

are already defined. Let

$$t_\alpha = p_\alpha\text{-}\lim_n w_{\alpha,n}, \tag{21}$$

which exists by compactness of $\mathbb{T}$.

Define $\rho : \mathbb{Z} \to \mathbb{T}$ by $\rho(k) = kt_\alpha$. The injectivity of $\mathbb{T}$ supplies an extension $\theta_\xi : \mathbb{Q} \to \mathbb{T}$. In particular,

$$\theta_\xi(1) = t_\alpha. \tag{22}$$

If $\xi \notin D \cup \iota[\mathfrak{c}]$, put $\theta_\xi = 0$.

The family $(\theta_\xi)_{\xi<\mathfrak{c}}$ induces a homomorphism on the direct sum $\mathbb{Q}^{(\mathfrak{c})}$, namely

$$\widetilde{\psi} : \mathbb{Q}^{(\mathfrak{c})} \longrightarrow \mathbb{T}, \qquad \widetilde{\psi}(z) = \sum_{\xi \in \operatorname{supp} z} \theta_\xi(z(\xi)). \tag{23}$$

Finally, let $\psi = \widetilde{\psi} \restriction G$.

We first verify that ψ extends ψ_D. If $z \in \mathbb{Q}^{(D)}$, then

$$\widetilde{\psi}(z) = \sum_{\xi \in \operatorname{supp} z} \theta_\xi(z(\xi)) = \sum_{\xi \in \operatorname{supp} z} \widetilde{\psi}_D(z(\xi)e_\xi) = \widetilde{\psi}_D(z).$$

Thus $\widetilde{\psi}$ extends $\widetilde{\psi}_D$, so its restriction ψ extends ψ_D.

We now verify the ultrafilter limit identities. Fix $\alpha < \mathfrak{c}$. Suppose first that $\iota(\alpha) \in D$. The hypothesis on D gives $\operatorname{supp} h_\alpha(n) \subseteq D$ for every $n \in \mathbb{N}$. Hence $h_\alpha(n), e_{\iota(\alpha)} \in G_D$, and, since ψ extends ψ_D, (16) gives

$$p_\alpha\text{-}\lim_n \psi(h_\alpha(n)) = \psi(e_{\iota(\alpha)}).$$

Suppose now that $\iota(\alpha) \notin D$. By (11),

$$\operatorname{supp} h_\alpha(n) \subseteq \iota(\alpha) \qquad (n \in \mathbb{N}),$$

so every coordinate occurring in $h_\alpha(n)$ precedes $\iota(\alpha)$. Consequently, (23) and (20) give

$$\psi(h_\alpha(n)) = \widetilde{\psi}(h_\alpha(n)) = w_{\alpha,n}.$$

Therefore, by (21) and (22),

$$p_\alpha\text{-}\lim_n \psi(h_\alpha(n)) = t_\alpha = \theta_{\iota(\alpha)}(1) = \psi(e_{\iota(\alpha)}).$$

□

Proposition 6.4 provides a homomorphism $\psi_D : G_D \to \mathbb{T}$ that does not vanish at the chosen nonzero vector x and satisfies (16). By Proposition 7.1, it extends to a homomorphism on G satisfying all the limit identities stated there.

Corollary 7.2. *For every $0 \neq x \in G$ there exists a homomorphism $\psi_x : G \to \mathbb{T}$ such that*

(i) $\psi_x(x) \neq 0$*;*
(ii) *for every $\alpha < \mathfrak{c}$,*

$$p_\alpha\text{-}\lim_n \psi_x(h_\alpha(n)) = \psi_x(e_{\iota(\alpha)}).$$

8. The group topologies

For each $0 \neq x \in G$ choose a homomorphism ψ_x as in Corollary 7.2. Define

$$\Delta_G : G \longrightarrow \mathbb{T}^{G\setminus\{0\}}, \qquad \Delta_G(z) = (\psi_x(z))_{x\neq 0},$$

and give G the initial topology τ_G induced by this map.

Lemma 8.1. *The topology τ_G is a Tychonoff group topology. For every $\alpha < \mathfrak{c}$, the free ultrafilter p_α satisfies*

$$p_\alpha\text{-}\lim_n h_\alpha(n) = e_{\iota(\alpha)}. \tag{24}$$

Proof. The initial topology of a homomorphism into a topological group is a group topology. The diagonal is injective: if $z \neq w$, then $x = z - w \neq 0$, and Corollary 7.2(i) gives $\psi_x(x) \neq 0$, so $\psi_x(z) \neq \psi_x(w)$. Since $\mathbb{T}$ is Hausdorff, the induced topology is Hausdorff. In fact, Δ_G identifies (G, τ_G) with a subspace of a power of the compact Hausdorff group $\mathbb{T}$, so (G, τ_G) is Tychonoff.

For a fixed $\alpha < \mathfrak{c}$, Corollary 7.2(ii) gives

$$p_\alpha\text{-}\lim_n \psi_x(h_\alpha(n)) = \psi_x(e_{\iota(\alpha)}) \qquad (x \neq 0).$$

Hence $\Delta_G(h_\alpha(n))$ converges along p_α to $\Delta_G(e_{\iota(\alpha)})$, since convergence in a product is coordinatewise. As τ_G is induced by the injective map Δ_G, this is precisely (24). $\square$

We are now ready to give the proofs of Theorem 1.1 and Proposition 1.2.

Proof of the main results. Lemma 8.1 shows that τ_G is a Hausdorff group topology. Let $s : \mathbb{N} \to G$ be injective. The enumeration in Section 5 gives a unique $\alpha < \mathfrak{c}$ with $s = s_\alpha$. By Lemma 5.1, the map $\varphi_\alpha : \mathbb{N} \to \mathbb{N}$ is strictly increasing and $h_\alpha = s \circ \varphi_\alpha$. By Lemma 8.1 and (24), the free ultrafilter p_α satisfies

$$p_\alpha\text{-}\lim_n s(\varphi_\alpha(n)) = e_{\iota(\alpha)} \neq 0.$$

Thus all the hypotheses of Lemma 2.2 hold, so (G, τ_G) is countably compact and every convergent sequence in it is eventually constant. This proves Theorem 1.1. Taking $G = \mathbb{Q}^{(\mathfrak{c})}$ and transporting the resulting topology along an algebraic isomorphism $\mathbb{Q}^{(\mathfrak{c})} \cong (\mathbb{R}, +)$ gives Proposition 1.2. $\square$

9. The Wallace semigroup

Specialize the preceding construction to $G = \mathbb{Z}^{(\mathfrak{c})}$, with its canonical basis, and let $\tau = \tau_G$ be the resulting topology. Consider

$$P = \{z \in \mathbb{Z}^{(\mathfrak{c})} : z(\xi) \geq 0 \text{ for every } \xi < \mathfrak{c}\} \subseteq \mathbb{Z}^{(\mathfrak{c})}.$$

Of course, P is a subsemigroup of $\mathbb{Z}^{(\mathfrak{c})}$ under addition, so it is two-sided cancellative and Tychonoff, and it is clearly not a group. It remains to see that P is countably compact.

Proof of Corollary 1.3. Let $A \subseteq P$ be countably infinite and choose an injective enumeration $s : \mathbb{N} \to A$. Let $\alpha < \mathfrak{c}$ be the unique index for which $s = s_\alpha$. The subsequence $h_\alpha = s_\alpha \circ \varphi_\alpha$ remains in A, and

$$p_\alpha\text{-}\lim_n h_\alpha(n) = e_{\iota(\alpha)}$$

in $\mathbb{Z}^{(\mathfrak{c})}$. Both the sequence and its limit lie in P, so the same convergence holds in the subspace topology. Since p_α is free and h_α is injective, $e_{\iota(\alpha)}$ is an accumulation point of A in P. Thus, P is countably compact. □

10. Further consequences and concluding remarks

We collect consequences that use the main theorem together with earlier reductions. We begin with an intrinsic strengthening of the topology constructed in Section 8.

Proposition 10.1. *Let G be a torsion-free Abelian group of cardinality $\mathfrak{c}$, endowed with the topology τ_G constructed in Section 8. If $A \subseteq G$ is infinite, then*

$$|\overline{A}| = \mathfrak{c}.$$

Consequently, every infinite closed subset of (G, τ_G) has cardinality $\mathfrak{c}$.

Proof. Choose a countably infinite set $B \subseteq A$. There are $\mathfrak{c}$ bijections from $\mathbb{N}$ onto B. Each of them occurs in the enumeration $(s_\alpha)_{\alpha<\mathfrak{c}}$ fixed in Section 5. For such an index α, the sequence $h_\alpha = s_\alpha \circ \varphi_\alpha$ takes its values in B, and Lemma 8.1 gives

$$p_\alpha\text{-}\lim_n h_\alpha(n) = e_{\iota(\alpha)}.$$

Since p_α is free, $e_{\iota(\alpha)} \in \overline{B}$. The map ι is injective, so the $\mathfrak{c}$ bijections from $\mathbb{N}$ onto B yield $\mathfrak{c}$ distinct points of $\overline{B}$. Hence

$$\mathfrak{c} \le |\overline{B}| \le |\overline{A}| \le |G| = \mathfrak{c}.$$

□

Shakhmatov asked whether there exists in ZFC a countably compact free Abelian group in which every infinite closed subset has cardinality at least $\mathfrak{c}$ [27, p. 13]. For $G = \mathbb{Z}^{(\mathfrak{c})}$, Proposition 10.1 shows that the topology constructed here provides such a group, thereby answering the question in ZFC.

The bicyclic semigroup $C(p, q)$ is the monoid generated by two elements p and q subject to the single relation $qp = 1$; every element of $C(p, q)$ has a unique representation $p^n q^m$ with $n, m \in \mathbb{N}$. Banakh, Dimitrova, and Gutik proved that the existence of a torsion-free Abelian countably compact topological group without nontrivial convergent sequences implies the existence of a Tychonoff countably compact topological semigroup containing a copy of the bicyclic semigroup [3, Theorem 6.6]. Their Problem 7.1 asks whether such a semigroup exists in ZFC.

Corollary 10.2. *In ZFC there exists a Tychonoff countably compact topological semigroup containing a copy of the bicyclic semigroup.*

Proof. Theorem 1.1, applied, for example, to $\mathbb{Z}^{(\mathfrak{c})}$, supplies the group required in [3, Theorem 6.6]. $\square$

Thus Problem 7.1 of [3] is answered in ZFC.

There is a parallel consequence for paratopological groups. Recall that a paratopological group is a group endowed with a topology for which multiplication is continuous. Guran asked whether there exists a Hausdorff countably compact paratopological group which is not a topological group and whether every Hausdorff countably compact paratopological group is totally bounded [16, Problems 1 and 2]. Banakh and Ravsky isolated the hypothesis TT: the existence of an infinite torsion-free Abelian countably compact topological group without nontrivial convergent sequences. Under TT, their Lemma 3.21 and Example 3.22 construct a functionally Hausdorff countably compact paratopological group which is not a topological group and whose underlying abstract group is free Abelian [4].

Corollary 10.3. *In ZFC there exists a functionally Hausdorff countably compact paratopological group which is not a topological group and whose underlying abstract group is free Abelian.*

Proof. Theorem 1.1 establishes TT in ZFC. Apply [4, Lemma 3.21 and Example 3.22]. $\square$

This answers both problems of Guran. Indeed, Banakh and Ravsky show as part of the construction that the example is not left-ω-precompact; in particular, it is not totally bounded [4, Example 3.22 and the discussion following it]. It also answers all three clauses of [4, Problem 3.23]: the example is Hausdorff and countably compact; it is Hausdorff, countably pracompact, and Baire; and it is T_1 and countably compact. In the terminology of [1, Problems 2.4.1 and 2.4.3], it gives a ZFC Hausdorff countably compact paratopological group which is not a topological group and is not precompact.

Another consequence is a monothetic refinement of the Wallace phenomenon. A topological monoid M is monothetic if it is topologically generated by one element, that is, if $M = \overline{\{a^n : n \in \mathbb{N}\}}$ for some $a \in M$. Banakh, Bardyla, Guran, Gutik, and Ravsky asked whether the local compactness assumption in their theorem on monothetic monoids can be weakened to local countable compactness. Under TT, they constructed a Hausdorff countably compact monothetic topological monoid which embeds into a topological group but is not a group [2, Remark 4.5].

Corollary 10.4. *In ZFC there exists a Hausdorff countably compact monothetic topological monoid which is a submonoid of a topological group but is not a group.*

Proof. Again Theorem 1.1 establishes TT, so the construction in [2, Remark 4.5] applies. $\square$

Finally, we record a consequence for *suitable sets*. A suitable set in a topological group is a discrete subset whose union with the identity is closed and which topologically generates the group. Dikranjan asked, in particular, whether there is a topologically finitely generated topological group without infinite suitable sets [10, Question 5.4].

Proposition 10.5. *In ZFC there exists an infinite monothetic countably compact topological group without nontrivial convergent sequences and without infinite suitable sets.*

Proof. Let (G, τ_G) be supplied by Theorem 1.1, choose $0 \neq g \in G$, and put $H = \overline{\langle g \rangle}$. The group H is infinite and monothetic: the cyclic group $\langle g \rangle$ is infinite because G is torsion-free. The subgroup H is closed in G, hence countably compact, and as a subspace of G it has no nontrivial convergent sequences.

Suppose that $S \subseteq H$ were an infinite suitable set and choose distinct points $s_n \in S$. We claim that $s_n \to 0$. Let U be an open neighborhood of 0. If infinitely many s_n lay outside U, their set would have an accumulation point in the closed countably compact subspace $H \setminus U$. Since $S \cup \{0\}$ is closed, that accumulation point would belong to $S \cup \{0\}$; it could not be 0, and it could not belong to S because S is discrete. This is a contradiction. Hence all but finitely many s_n belong to U, proving the claim. The resulting nontrivial convergent sequence contradicts the defining property of G. Therefore H has no infinite suitable set. □

Proposition 10.5 answers the example clause of [10, Question 5.4] in ZFC.

Acknowledgments

The second author was supported by the São Paulo Research Foundation (FAPESP), grant no. 2025/07302-0.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

OpenAI Codex, powered by the GPT-5.6 Sol model, was used extensively as a generative research and writing tool. In an iterative process directed by the authors, the model carried out most of the exploratory proof development and generated most of the first draft of the manuscript and the accompanying Lean 4 formalization. The authors formulated the problem and objectives, provided the mathematical context and constraints, steered the iterations through prompts and feedback, supplied ideas and suggested strategies, selected and revised the outputs, reviewed the final arguments, and extensively rewrote the text, using AI again for proofreading and review. The authors take full responsibility for the accuracy, originality, and integrity of the work.

Dedication

The authors dedicate this paper to Professor Artur Hideyuki Tomita (University of São Paulo, Brazil), who supervised them during their doctoral studies. The authors' earlier research under his guidance provided invaluable preparation for the present work. Professor Tomita made significant contributions to the problems studied in this paper, and many of the arguments presented here are inspired by his previous work. Without Professor Tomita's work, the present paper would not have been possible.

References

1. Alexander Arhangel'skii and Mikhail Tkachenko, *Topological groups and related structures*, Atlantis Studies in Mathematics, vol. 1, Atlantis Press/World Scientific, Amsterdam–Paris, 2008, doi:10.2991/978-94-91216-35-0.
2. Taras Banakh, Serhii Bardyla, Igor Guran, Oleg Gutik, and Alex Ravsky, *Positive answers to Koch's problem in special cases*, Topol. Algebra Appl. **8** (2020), no. 1, 76–87, doi:10.1515/taa-2020-0007.
3. Taras Banakh, Svetlana Dimitrova, and Oleg Gutik, *Embedding the bicyclic semigroup into countably compact topological semigroups*, Topology Appl. **157** (2010), no. 18, 2803–2814, doi:10.1016/j.topol.2010.08.020.
4. Taras Banakh and Alex Ravsky, *On feebly compact paratopological groups*, Topology Appl. **284** (2020), 107363, doi:10.1016/j.topol.2020.107363.
5. M. K. Bellini, K. P. Hart, V. O. Rodrigues, and A. H. Tomita, *Countably compact group topologies on arbitrarily large free Abelian groups*, Topology Appl. **333** (2023), 108538, doi:10.1016/j.topol.2023.108538.
6. M. K. Bellini, V. O. Rodrigues, and A. H. Tomita, *Forcing a classification of non-torsion Abelian groups of size at most* $2^{\mathfrak{c}}$ *with non-trivial convergent sequences*, Topology Appl. **296** (2021), 107684, doi:10.1016/j.topol.2021.107684.
7. A. C. Boero, I. Castro-Pereira, and A. H. Tomita, *Countably compact group topologies on the free Abelian group of size continuum (and a Wallace semigroup) from a selective ultrafilter*, Acta Math. Hungar. **159** (2019), no. 2, 414–428, doi:10.1007/s10474-019-00991-w.
8. W. W. Comfort, *Problems on topological groups and other homogeneous spaces*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 313–347.
9. D. Dikranjan and D. Shakhmatov, *Forcing hereditarily separable compact-like group topologies on Abelian groups*, Topology Appl. **151** (2005), no. 1–3, 2–54, doi:10.1016/j.topol.2004.07.012.
10. Dikran Dikranjan, *Generators of topological groups*, General and Geometric Topology (Kyoto) (Kazuhiro Kawamura, ed.), RIMS Kôkyûroku, vol. 1074, Research Institute for Mathematical Sciences, Kyoto University, 1999, RIMS Kôkyûroku 1074, pp. 102–125.
11. R. Engelking, *General topology*, revised and completed ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann, Berlin, 1989.
12. L. Fuchs, *Infinite abelian groups*, vol. I, Pure and Applied Mathematics, no. 36, Academic Press, New York–London, 1970.
13. J. L. J. Fuentes-Maguiña, V. O. Rodrigues, and A. H. Tomita, *A Wallace semigroup whose every finite power is countably compact*, 2024, Preprint, arXiv:2403.00205.
14. B. R. Gelbaum, G. K. Kalisch, and J. M. H. Olmsted, *On the embedding of topological semigroups and integral domains*, Proc. Amer. Math. Soc. **2** (1951), 807–821, doi:10.1090/S0002-9939-1951-0043109-6.

15. D. L. Grant, *Sequentially compact cancellative topological semigroups: some progress on the Wallace problem*, Ann. New York Acad. Sci. **704** (1993), 150–154, doi:10.1111/j.1749-6632.1993.tb52518.x.
16. I. Guran, *Some open problems in the theory of topological groups and semigroups*, Matematychni Studii **10** (1998), no. 2, 223–225, Journal PDF.
17. M. Hrušák and A. Shibakov, *Convergent sequences in topological groups*, Ann. Pure Appl. Logic **172** (2021), no. 5, 102910, doi:10.1016/j.apal.2020.102910.
18. M. Hrušák, J. van Mill, U. A. Ramos-García, and S. Shelah, *Countably compact groups without non-trivial convergent sequences*, Trans. Amer. Math. Soc. **374** (2021), no. 2, 1277–1296, doi:10.1090/tran/8222.
19. Y. Katznelson, *An introduction to harmonic analysis*, 3 ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004, doi:10.1017/CBO9781139165372.
20. P. B. Koszmider, A. H. Tomita, and S. Watson, *Forcing countably compact group topologies on a larger free Abelian group*, Topology Proc. **25** (2000), 563–574, Topology Proceedings reprint.
21. R. E. Madariaga-Garcia and A. H. Tomita, *Countably compact topological group topologies on free Abelian groups from selective ultrafilters*, Topology Appl. **154** (2007), no. 7, 1470–1480, doi:10.1016/j.topol.2006.03.028.
22. A. Mukherjea and N. A. Tserpes, *A note on countably compact semigroups*, J. Austral. Math. Soc. **13** (1972), no. 2, 180–184, doi:10.1017/S1446788700011253.
23. K. Numakura, *On bicompact semigroups*, Math. J. Okayama Univ. **1** (1952), 99–108, doi:10.18926/mjou/33720.
24. E. Pearl, *Open problems in topology*, Topology Appl. **136** (2004), no. 1–3, 37–85, doi:10.1016/S0166-8641(03)00183-4.
25. D. Robbie and S. Svetlichny, *An answer to A. D. Wallace's question about countably compact cancellative semigroups*, Proc. Amer. Math. Soc. **124** (1996), no. 1, 325–330, doi:10.1090/S0002-9939-96-03418-1.
26. W. Rudin, *Principles of mathematical analysis*, 3 ed., International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1976.
27. D. Shakhmatov, *Recent progress in topological groups: selected topics*, RIMS Kôkyûroku **1188** (2001), 1–19, RIMS Kôkyûroku 1188.
28. M. G. Tkachenko, *Countably compact and pseudocompact topologies on free Abelian groups*, Soviet Math. (Iz. VUZ) **34** (1990), no. 5, 79–86, English translation of Izv. Vyssh. Uchebn. Zaved. Mat. 1990, no. 5, 68–75.
29. ______, *Topological groups: between compactness and $\aleph_0$-boundedness*, Recent Progress in General Topology II (M. Hušek and J. van Mill, eds.), North-Holland, Amsterdam, 2002, pp. 515–543.
30. A. H. Tomita, *The Wallace problem: a counterexample from* $\mathrm{MA}_{\mathrm{countable}}$ *and p-compactness*, Canad. Math. Bull. **39** (1996), no. 4, 486–498, doi:10.4153/CMB-1996-057-6.
31. ______, *Square of countably compact groups without non-trivial convergent sequences*, Topology Appl. **153** (2005), no. 1, 107–122, doi:10.1016/j.topol.2005.01.026.
32. ______, *A van Douwen-like ZFC theorem for small powers of countably compact groups without non-trivial convergent sequences*, Topology Appl. **259** (2019), 347–364, doi:10.1016/j.topol.2019.02.040.
33. A. H. Tomita and J. Trianon-Fraga, *Some pseudocompact-like properties in certain topological groups*, Topology Appl. **314** (2022), 108111, doi:10.1016/j.topol.2022.108111.
34. E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), no. 2, 417–427, doi:10.1090/S0002-9947-1980-0586725-8.
35. A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. **61** (1955), 95–112, doi:10.1090/S0002-9904-1955-09895-1.

Independent researcher, São Paulo, SP, Brazil
Email address: julianetrianon@gmail.com

Department of Mathematics, Institute of Mathematics, Statistics and Computer Science, University of São Paulo, Rua do Matão, 1010, 05508-090 São Paulo, SP, Brazil
Email address: vinior@ime.usp.br